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A note on Bernstein-Jordan Algebras (2)

Kohei Miyamoto

Department of Education, Faculty of Letters
Mukogawa Women's University, Nishinomiya 663-8558, Japan

Abstract

As a contribution of the problem of classifying all Bernstein algebras, we attempt to describe the posibility of u-part decomposition of Bernstein-Jordan algebras in terms of the direct product of Bernstein algebras and study some conditions for a Bernstein-Jordan algebra to be u-part decomposable.

§ 1. Preliminaries

Throughout this paper we shall consider commutative algebras of finite dimension over an infinite field K of characteristic not two.

A commutative algebra A over K is called a *Bernstein algebra* if there exists a nonzero algebra homomorphism $\omega : A \to K$ (ω is called *weight homomorphism*) and every element x in A satisfies the equation

$$(x^2)^2 = \omega(x)^2 x^2.$$

The following is a brief description of some known results on Bernstein algebras, which will be used in this study.

(i) For any Bernstein algebra the weight homomorphism is uniquely determined.

(ii) Any Bernstein algebra A has at least one idempotent. For each idempotent $e \in A$, one has $A = Ke \oplus U_e \oplus Z_e$ (direct sum of vector spaces) called the *Peirce decomposition* of A associated to e, where

$$U_e = \{x \in A \mid ex = 1/2x\}, \quad Z_e = \{x \in A \mid ex = 0\}$$

and $\mathrm{Ker}\,\omega = U_e + Z_e$. Then the relations

$$U_e Z_e \subseteq U_e, \quad Z_e^2 \subseteq U_e, \quad U_e^2 \subseteq Z_e \tag{1}$$

and the identities

$$u^3 = u(uz) = uz^2 = (uz)^2 = (u^2)^2 = 0 \tag{2}$$

for every $u \in U_e$ and $z \in Z_e$ hold.

For any given idempotent e of A, the set of idempotent elements of A is given by $E(A) = \{e + t + t^2 \mid t \in U_e\}$, and for two idempotents e and $f = e + t + t^2$, we have the relations

$$U_f = \{u + 2tu \mid u \in U_e\} \tag{3}$$
$$Z_f = \{z - 2(t + t^2)z \mid z \in Z_e\} \tag{4}$$

between the corresponding Peirce spaces, and it is also known that $\dim U_e = \dim U_f$ and $\dim Z_e = \dim Z_f$. A Bernstein algebra A is said to be *trivial* if $U_e = Z_e = \langle 0 \rangle$, that is, $A \cong K$. The Peirce spaces U_e and Z_e associated to an idempotent e may be written simply U and Z, respectively, when e is fixed.

By definition a commutative algebra A is called a Jordan algebra if the identity $x^2(xy) = (x^2 y)x$ holds for every x, y in A. We will call a commutative algebra A a *Bernstein-Jordan algebra* if A is both Bernstein algebra and Jordan algebra.

(iii) A Bernstein algebra $A=Ke\oplus U_e\oplus Z_e$ with an idempotent e is a Jordan algebra if and only if $z^2=(uz)z=0$ for every element u in U_e and z in Z_e. Furthermore, a Bernstein algebra A is Jordan if and only if $Z_f^2=\langle 0\rangle$ for every idempotent f of A(cf. [2]).

(iv) If A is a Bernstein-Jordan algebra(of finite dimension), the ideal $\mathrm{Ker}\omega$ is nilpotent, that is, there exists a positive integer m such that $(\mathrm{Ker}\omega)^{m+1}=\langle 0\rangle$.

(v) If $A=Ke\oplus U_e\oplus Z_e$ is a Bernstein-Jordan algebra, then the subspace $I_e=U_eZ_e\oplus Z_e$ of A is a *baric* ideal, that is, ideal included in $\mathrm{Ker}\omega$. Moreover, $I_e=I_f$ always holds for any two idempotents e and f of A(cf. [3]). We denote this ideal by I. The factor space A/I is a Bernstein-Jordan algebra and the weight homomorphism $\overline{\omega}$ of A/I is defined by $\overline{\omega}(\phi(x))=\omega(x)$ for any element x in A, where ϕ is the canonical homomorphism of A onto A/I.

§2. Decomposability of Bernstein-Jordan algebras

As shown by T. Cortés and F. Montaner(cf. [1]), the *direct product* of a family of Bernstein algebras $(A_i,\ \omega_i)_{1\leq i\leq n}$ is a Bernstein algebra $\times_{1\leq i\leq n} A_i$, which is defined as a set by

$$\times_{1\leq i\leq n} A_i=$$

$$\{(x_i)\in \prod_{1\leq i\leq n} A_i \mid \omega_i(x_i)=\omega_j(x_j) \text{ for all } i,\ j\}$$

and, as an algebra, is a subalgebra of $\Pi_{1\leq i\leq n} A_i$ equipped with the weight homomorphism ω defined by $\omega((x_i))=\omega_j(x_j)$ for any $j(1\leq j\leq n)$. This implies that for any given Bernstein algebra C with the weight homomorphism τ and homomorphisms $\varphi_i : C\to A_i$ such that $\tau=\omega_i\varphi_i$ for all $i(1\leq i\leq n)$, there always exists the unique homomorphism $\varphi : C\to \times A_i$ such that $\omega\varphi=\tau$.

Then, by definition, a Bernstein algebra A is *decomposable* if there exist nontrivial Bernstein algebras A_1 and A_2 such that $A\cong A_1\times A_2$ and *indecomposable* if it is not decomposable.

It is known that the decomposability of Bernstein algebras is characterized as follows(cf. [1]).

Proposition A. A Bernstein algebra A is decomposable if and only if there exist nonzero baric ideals I_1, I_2 of A such that $\mathrm{Ker}\omega=I_1\oplus I_2$ as algebras. In this case $A\cong A/I_1\times A/I_2$.

Proposition B. A Bernstein algebra A is decomposable if and only if there exist nontrivial subalgebras A_1, A_2 of A such that $(A_1\cap \mathrm{Ker}\omega)(A_2\cap \mathrm{Ker}\omega)=\langle 0\rangle$, $A_1+A_2=A$, and $A_1\cap A_2$ is a trivial algebra. In this case $A\cong A_1\times A_2$.

Remark 1. The ideals I_i and subalgebras A_i in propositions stated above are related by

$$A/I_1\cong A_2,\ \ A/I_2\cong A_1,\ \ A_i=Ke+I_i.$$

T. Cortés and F. Montaner reduced the problem of classifing all Bernstein algebras to the case of indecomposable Bernstein-Jordan algebras, while they found that there exist infinite families of indecomposable Bernstein-Jordan algebras(cf. [1]). Therefore it is expected that the decomposition of indecomposable Bernstein-Jordan algebras into Bernstein-Jordan algebras of more simple type.

Now we shall define *u-part decomposability* for Bernstein-Jordan algebras.

Definition 2. Let A be a Bernstein-Jordan algebra A such that $U_e\neq\langle 0\rangle$ for any idempotent e of A. If there exist nonzero subspaces U_1 and U_2 of U_e such that

$$U_e=U_1\oplus U_2,\ \text{and}\ \ U_iZ_e\subset U_i \tag{5}$$

for $i=1$, 2, then A is said *u-part decomposable.*

We can prove that u-part decomposability is not dependent on the choice of idempotent e(cf.

[3]).

Let $A=Ke\oplus U\oplus Z$ be the Peirce decomposition of a Bernstein algebra A associated to some idempotent e and $\phi: A\rightarrow A/I$ be the canonical homomorphism where the ideal $I=UZ\oplus Z$(see § **1**(v)). Then we can put $\phi(x)\equiv\alpha e+u(\text{mod } I)$ for all $x=\alpha e+u+z\in A$.

Lemma 3.

(a) For the factor algebra A/I, there exists an isomorphism $A/I\cong Ke\oplus U/UZ$ as spaces.

(b) If A is u-part decomposable and U_1, U_2 are both nonzero subspaces of U satisfying the condition (5), then $U/UZ\cong U_1/U_1Z\oplus U_2/U_2Z$ as spaces.

Proof. (a) Since $I=UZ\oplus Z$ is a baric ideal of A and UZ is a subspace of U, $A/I\cong Ke\oplus U/UZ\oplus Z/Z\cong Ke\oplus U/UZ$.

(b) Since $U_iZ\subset U_i$ and $U_1\cap U_2=\langle 0\rangle$, we have $U_1Z\cap U_2Z=\langle 0\rangle$, so $UZ=U_1Z\oplus U_2Z$. Therefore $U/UZ\cong U_1/U_1Z\oplus U_2/U_2Z$. □

Remark 4. If we define spaces $A_i=Ke\oplus U_i\oplus Z$ and $\overline{A_i}=A_i/(A_i\cap I)\,(i=1,\ 2)$, then, as easily seen, they are Bernstein algebras with weight homomorphisms $\omega_i=\omega\mid_{A_i}$ and $\overline{\omega_i}=\overline{\omega}\mid_{\overline{A_i}}$, respectively, and $\overline{A_i}\cong Ke\oplus U_i/(U_iZ)$ as spaces.

In the following discussion we assume that algebra A is a Bernstein-Jordan algebra with the weight homomorphism ω such that $\dim U>0$. Let $A=Ke\oplus U\oplus Z$ be the Peirce decomposition of A associated to some idempotent e. We recall that I is the (baric) ideal defined by $I=UZ\oplus Z$.

We assert that, if A is u-part decomposable, then the factor algebra A/I is decomposable.

Proposition 5. Let A be u-part decomposable and $U_i(i=1,\ 2)$ nonzero subspaces such that $U=U_1\oplus U_2$ and $U_iZ\subset U_i$, and A_i Bernstein-Jordan algebras defined by $A_i=Ke\oplus U_i\oplus Z(i=1,\ 2)$. Then $\overline{A}\cong\overline{A_1}\times\overline{A_2}$ where $\overline{A}=A/I$ and $\overline{A_i}=A_i/(A_i\cap I)\,(i=1,\ 2)$.

Proof. For $i=1,\ 2$ let π_i be the canonical projection of U onto U_i and define the mapping σ_i of $\overline{A}$ to $\overline{A_i}$ by $\sigma_i(\phi(x))=\alpha e+\phi(\pi_i(u))$ for any $x=\alpha e+u+z\in A$, where ϕ is the canonical homomorphism $A\rightarrow\overline{A}$. Then we can see easily that $\sigma_i(i=1,\ 2)$ are well-defined homomorphisms. We show that σ_i satisfies $\overline{\omega}=\overline{\omega_i}\sigma_i$ for $i=1,\ 2$. For any element $x=\alpha e+u+z$ of A we have $\overline{\omega}(\phi(x))=\omega(x)=\alpha$ and $\overline{\omega_i}\sigma_i(\phi(x))=\overline{\omega_i}(\alpha e+\phi(\pi_i(u)))=\omega_i(\alpha e+\pi_i(u))=\alpha$. Hence $\overline{\omega}=\overline{\omega_i}\sigma_i$.

We define the mapping σ of $\overline{A}$ to $\overline{A_1}\times\overline{A_2}$ by $\sigma(\phi(x))=(\sigma_1(\phi(x)),\ \sigma_2(\phi(x)))$ for each $x\in A$. Then we can prove that σ gives an isomorphism of $\overline{A}$ onto $\overline{A_1}\times\overline{A_2}$ as follows.

First we note that σ is well-defined since $\overline{\omega_1}\sigma_1(\phi(x))=\overline{\omega_2}\sigma_2(\phi(x))=\overline{\omega}(x)$ for any $x\in A$. By easy calculation we see that σ is a homomorphism of algebras. Let $\sigma(\phi(x))=0$ for $x=\alpha e+u+z\in A$. Then $\sigma_i(\phi(x))=\alpha e+\phi(\pi_i(u))\equiv 0(\text{mod } A_i\cap I)$, so $\alpha=0$, $\phi(\pi_i(u))=UZ$ and $\pi_i(u)\subseteq UZ\cap U_i$ for $i=1,\ 2$. Therefore $u=\pi_1(u)+\pi_2(u)\in UZ$, so $\phi(u)\equiv 0(\text{mod } I)$ and $\phi(x)\equiv 0(\text{mod } I)$, which means that σ is a monomorphism. It is trivial that ϕ is an epimorphism. □

We concern on the inverse problem of Proposition 5.

Assume that $\overline{A}=A/I$ with $I=UZ\oplus Z$ and $\overline{A}$ is decomposable. Then, by Proposition A, there exist nonzero baric ideals $\overline{I_i}(i=1,\ 2)$ of $\overline{A}$ such that $\text{Ker}\overline{\omega}=\overline{I_1}\oplus\overline{I_2}$ and in this case $\overline{A}\cong\overline{A}/\overline{I_1}\times\overline{A}/\overline{I_2}$. Let $\psi_i:\overline{A}\rightarrow\overline{A}/\overline{I_i}(i=1,\ 2)$ be the canonical homomorphisms, τ_i the weight homomorphism of $\overline{A}/\overline{I_i}$ defined by $\tau_i(\psi_i(\phi(x)))=\overline{\omega}(\phi(x))$ and τ the weight homomorphism of $\overline{A}/\overline{I_1}\times\overline{A}/\overline{I_2}$ defined by $\tau(\psi_1(\phi(x)),\ \psi_2(\phi(x)))=\tau_i(\psi_i(\phi(x)))$. Then the isomorphism $\psi:\overline{A}\cong\overline{A}/\overline{I_1}\times\overline{A}/\overline{I_2}$ is defined by

$$\psi(\phi(x))=(\psi_1\phi(x),\ \psi_2\phi(x))$$

for $\phi(x)\in\overline{A}$.

Moreover define the epimorphisms $\phi_i: A\rightarrow\overline{A}/$

$\overline{I}_i(i=1, 2)$ by

$$\phi_i=\psi\phi$$

and the subspaces $I_i(i=1, 2)$ of $\mathrm{Ker}\omega$ by

$$I_i=\phi^{-1}(\overline{I}_i).$$

Lemma 6. The space I_i satisfies $I_i=\mathrm{Ker}\phi_i$ for $i=1, 2$

proof. Let x be any element of A. Since $\phi_i(x)=\psi_i\phi(x)\equiv\phi(x)\ (\mathrm{mod}\ \overline{I}_i)$, the condition $x\in\mathrm{Ker}\phi_i$ is equivalent to the condition $\phi(x)\in\overline{I}_i$, i.e. $x\in I_i$. □

Lemma 7. The space I_i is a baric ideal of A satisfying $I_iZ\subseteq I_i$ for $i=1, 2$.

proof. For any $x\in I_i$ and $z\in Z$ we obtain that $\phi(xz)=\phi(x)\phi(z)\in\overline{I}_i$ since $\overline{I}_i$ is an ideal, which implies $I_iZ\subseteq I_i$. We note that $\psi\phi(I_1)=(\overline{I}_1, \overline{I}_1+\overline{I}_2)$, so $\phi_1(I_1)=\overline{I}_1$, which means that $I_1\subseteq J_1$. Similarly $I_2\subseteq J_2$ is obtained. □

Now define the subspaces $V_i\ (i=1, 2)$ of U by

$$V_i=I_i\cap U.$$

Lemma 8. We have the following relations.

(a) $I_i=V_i\oplus Z(i=1, 2)$

(b) $V_iZ\subseteq V_i(i=1, 2)$

(c) $V_1\cap V_2=UZ$

(d) $U=V_1+V_2$

proof. (a) For any element z of Z and $i=1, 2$, $\phi_i(z)=\psi_i\phi(z)=\psi_i(0+I)\equiv 0(\mathrm{mod}\ \overline{I}_i)$ since $\phi(z)\equiv 0(\mathrm{mod}\ I)$, so $z\in\mathrm{Ker}\phi_i$. This implies $Z\subseteq I_i$ by Lemma 6. Hence $I_i=I_i\cap(U\oplus Z)=I_i\cap U\oplus I_i\cap Z$ and $I_i\cap Z=Z$. By definition of V_i, we obtain $I_i=V_i\oplus Z$ as spaces. (b) $V_iZ=(I_i\cap U)Z\subseteq I_iZ\cap UZ\subseteq I_i\cap U=V_i$ by Lemma 7. (c) Let x be any element of $V_1\cap V_2$. From the definition of V_i, $x\in U\cap V_1\cap V_2$. Then we have $\phi(x)\in\overline{I}_1\cap\overline{I}_2$ since $\phi_i(x)\equiv\phi(x)\equiv 0(\mathrm{mod}\ \overline{I}_i)$ for $i=1, 2$ by Lemma 6.Therefore $\phi(x)\equiv 0(\mathrm{mod\ I})$, so $x\in I$. On the otherhand, since $x\in U$ and u-part of I is UZ, we have $x\in UZ$. As a result we obtain $V_1\cap V_2\subseteq UZ$. The inverse inclusion is easy. (d) For each $u\in U$ there exist uniquely elements $\overline{u}_i$ of $\overline{I}_i(i=1, 2)$ such that $\phi(u)=\overline{u}_1+\overline{u}_2$, since $\overline{\omega}(\phi(u))=\omega(u)=0$. Then we can take elements $u_i\in I_i(i=1, 2)$ such that $\phi(u_i)=\overline{u}_i$. Since for all elements z of Z $\phi(u_i+z)=\phi(u_i)$, we can suppose $u_i\in U(i=1, 2)$. Hence $u_i\in V_i$. Then by the way of choise of u_i, we have $\phi(u-u_1-u_2)=\phi(u)-\phi(u_1)-\phi(u_2)\equiv 0(\mathrm{mod}\ I)$, so $u-u_1-u_2\in I$. Hence $u-u_1-u_2\in I\cap U=UZ$, so $u\in UZ+V_1+V_2$. This means that $U\subseteq UZ+V_1+V_2$. On the otherhand $UZ\subseteq V_1+V_2$ from the result of (c) above. Consequently we obtain $U\subseteq V_1+V_2$. The inverse inclusion is obvious. □

For each $i=1, 2$ there exists a subspace W_i of V_i such that

$$V_i=UZ\oplus W_i.$$

For such W_i we define the subspace U_i of U by

$$U_i=W_i+\sum_{k\geq 1}W_iZ^{(k)},$$

where $W_iZ^{(k)}$ is defined by $W_iZ^{(1)}=W_iZ$, $W_iZ^{(k+1)}=(W_iZ^{(k)})Z$ recursively for each integer $k\geq 1$. We note that the summation above is a finite sum since $\mathrm{Ker}\omega=U+Z$ is nilpotent(cf. §**1**(iv))

Lemma 9. We have the following relations.

(a) $U_i\subseteq V_i$ and $U_iZ\subseteq U_i(i=1, 2)$

(b) $U_i=W_i\oplus\Sigma_{k\geq 1}W_iZ^{(k)}$

(c) $U=U_1+U_2$

(d) $UZ=\Sigma_{k\geq 1}W_1Z^{(k)}+\Sigma_{k\geq 1}W_2Z^{(k)}$

proof. (a) Since $W_i\subseteq V_i$, $U_i\subseteq V_i$ and $U_iZ\subseteq U_i$. (b) From $W_iZ^{(k)}\subseteq UZ$, it follows that $\Sigma W_iZ^{(k)}\subseteq UZ$. Since $UZ\cap W_i=\langle 0\rangle$, it follows that $\Sigma W_iZ^{(k)}\cap W_i=\langle 0\rangle$. (c) From the equation $U=V_1+V_2=UZ+W_1+W_2$ we obtain by induction $U\subseteq UZ^{(m)}+U_1+U_2$ for all integer $m\geq 1$. When m is large enough, $UZ^{(m)}=\langle 0\rangle$(cf. §**1**(iv)). Therefore $U \subseteq U_1+U_2$, so $U= U_1+U_2$. (d) By the similar fashon as the proof of (c) above we have $UZ \subseteq UZ^{(m)}+\Sigma W_1Z^{(k)}+\Sigma W_2Z^{(k)}$ for every integer $m \geq 1$. Thus we obtain $UZ \subseteq \Sigma W_1Z^{(k)}+\Sigma W_2Z^{(k)}$. The inverse inclusion is obvious. □

Proposition 10. If $\Sigma_{k\geq 1}^{m} W_1Z^{(k)} \cap \Sigma_{k\geq 1}^{m} W_2Z^{(k)} = \langle 0 \rangle$, then the spaces $U_i=W_i+\Sigma_{k\geq 1}^{m} W_iZ^{(k)} (i=1, 2)$ satisfies that $U=U_1 \oplus U_2$ and $U_iZ \subseteq U_i$, that is, a Bernstein-Jordan algebra A is u-part decomposable.

proof. From the results of Lemma 8 (d) and Lemma 9 (d) we see that $U=V_1+V_2=UZ+W_1+W_2=\Sigma W_1Z^{(k)}+\Sigma W_2Z^{(k)}+W_1+W_2$. We can show directly that $\Sigma W_1Z^{(k)}+\Sigma W_2Z^{(k)}+W_1+W_2$ is direct sum of spaces as follows. By hyposesis of the proposition, $\Sigma W_1Z^{(k)} \cap \Sigma W_2Z^{(k)}=\langle 0 \rangle$. Since $UZ \cap W_1=\langle 0 \rangle$, $(\Sigma W_1Z^{(k)}+\Sigma W_2Z^{(k)}) \cap W_1=\langle 0 \rangle$. Then, since $V_1 \cap W_2 \subseteq UZ$ by Lemma 8 (c) and $UZ \cap W_2=\langle 0 \rangle$, it follows that $(\Sigma W_1 Z^{(k)}+\Sigma W_2Z^{(k)}+W_1) \cap W_2=(UZ+W_1) \cap W_2=V_1 \cap W_2=\langle 0 \rangle$. These imply that $\Sigma W_1Z^{(k)}+\Sigma W_2Z^{(k)}+W_1+W_2=\Sigma W_1Z^{(k)} \oplus \Sigma W_2Z^{(k)} \oplus W_1 \oplus W_2$. Hence $U=(W_1 \oplus \Sigma W_1Z^{(k)}) \oplus (W_2 \oplus \Sigma W_1Z^{(k)})=U_1 \oplus U_2$. □

Corollary 11. If $V_1Z \cap V_2Z=\langle 0 \rangle$, then A is u-part decomposable.

proof. Take a subspace W_i such that $V_i=UZ \oplus W_i$ for $i=1, 2$. By Lemma 8 (b), $W_iZ \subseteq V_iZ \subseteq V_i$. Therefore, by induction, $W_iZ^{(k)} \subseteq V_iZ$ for each integer $k \geq 1$, so $\Sigma W_iZ^{(k)} \subseteq V_iZ$. Hence $\Sigma W_1Z^{(k)} \cap \Sigma W_2Z^{(k)} \subseteq V_1Z \cap V_2Z=\langle 0 \rangle$. The rest of proof is obvious from Proposition 10. □

Theorem 12. Assume that $A=Ke+U+Z$ is a nontrivial Bernstein-Jordan algebra and I is the ideal of A defined by $I=UZ \oplus Z$ such that $\overline{A}=A/I$ is decomposable with $I^2=\langle 0 \rangle$. Then A is u-part decomposable, that is, there exist nonzero subspaces $U_i(i=1, 2)$ such that $U=U_1 \oplus U_2$ and $U_iZ \subset U_i$.

proof. First we observe that the condition $I^2=\langle 0 \rangle$ is equivalent to that $(UZ)Z=\langle 0 \rangle$ since $Z^2=\langle 0 \rangle$ and $(UZ)^2=\langle 0 \rangle$ by the assumption and the properties of Bernstein-Jordan algebras. Now we take the bases $\{x_i; v_j\}_{1\leq i\leq r; 1\leq j\leq s}$ and $\{x_i; w_k\}_{1\leq i\leq r; 1\leq k\leq t}$ of V_1 and V_2 respectively such that $\{x_i\}_{1\leq i\leq r}$ is the basis of UZ, and $\{v_j\}_{1\leq j\leq s}$ and $\{w_k\}_{1\leq k\leq t}$ are the bases of W_1 and W_2 respectively, where $r=\dim UZ$, $s=\dim W_1$, and $t=\dim W_2$. Then we can assume that there exists at least one base element v in the set $B=\{v_j; w_k\}_{1\leq j\leq s; 1\leq k\leq t}$ such that $vZ \neq \langle 0 \rangle$. Because, if it is not so, $V_1Z=V_2Z=\langle 0 \rangle$ from $(UZ)Z=\langle 0 \rangle$, so the proof is reduced to Corollary 11. We separate the set B into two nonempty sets $B_1=\{y \in B \mid yZ \neq \langle 0 \rangle\}$, $B_2=\{y \in B \mid yZ=\langle 0 \rangle\}$ and construct subspaces V'_i, $(i=1, 2)$ of U, where $V'_i(i=1, 2)$ are respectevely defined as the spaces spanned by the sets $\{x_i\}_{1\leq i\leq r} \cup B_i$. Then, as easily seen, $U=V'_1+V'_2$, $V'_iZ \subseteq V'_i$, $V'_1 \cap V'_2=UZ$ and $V'_1Z \cap V'_2Z=V'_1Z \cap \langle 0 \rangle=\langle 0 \rangle$. Hence the proof is reduced to Corollary 11 again. □

Example 13. As an example of indecomposable but u-part decomposable algebra, we take up the Bernstein-Jordan algebra $A=A(\alpha, \beta)$ for $\alpha, \beta \in K$, with basis $e, u_1, u_2, u_3, u_4, z_1, z_2$ and multiplication table $e^2=e$, $eu_i=1/2u_i$, $ez_i=0$, $u_1^2=z_1$, $u_2^2=z_1+z_2$, $u_3^2=z_1+\alpha z_2$, $u_4^2=z_1+\beta z_2$, $u_iu_j=0$ for $i \neq j$ and $z_iz_j=u_iz_j=0$. It is known that this algebra A is indecomposable and that there exist infinitely many nonisomorphic algebras of the form $A(\alpha, \beta)$ (cf. [1]). In this case the ideal $I=UZ+Z$ is equal to $Z=Kz_1+Kz_2$ since $UZ=\langle 0 \rangle$ by multiplication table. Moreover $\overline{A}=A/I \cong Ke \oplus \overline{I_1} \oplus \overline{I_2} \oplus \overline{I_3} \oplus \overline{I_4}$, where $\overline{I_i} \cong Ku_i$ are nonzero baric ideals. Hence $\overline{A}$ is decomposable by Proposition A. Therefore A is u-part decomposable by Theorem 11 since $I^2=\langle 0 \rangle$.

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